Synchronizability of weighted aging scale-free networks

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(Received 15 June 2006; published 9 October 2006)

We study the synchronizability of weighted aging scale-free networks with non-normalized and asymmetrical coupling matrices. We found that the synchronizability of such networks is improved when the couplings from older to younger nodes become dominant, where the out degrees of the nodes are heterogeneous and their in degrees are homogeneous, and that the synchronizabilty of the networks is seriously weakened or even lost when the couplings from younger to older nodes become dominant, where the out degrees of the nodes are homogeneous and their in degrees are heterogeneous. We also found that both the heterogeneity of nodes and a smaller average degree can improve the synchronizability of the networks with some appropriately chosen weighting parameter values. We finally show an example of the coupled Lorenz systems for illustration and verification of the theoretical analysis.

DOI: 10.1103/PhysRevE.74.046107 PACS number(s): 89.75.Hc, 05.45.Xt, 89.75.−k

I. INTRODUCTION

Many social, biological, and communication systems can be cast into the form of complex networks. Various models have been presented and investigated recently [1–5], among which the statistical characteristics and dynamical performances of small-world network models (with near Poisson degree distributions) and scale-free network models (with power-law degree distributions) were studied quite intensively. Synchronizability is an important subject in the study of dynamical performance of complex networks. The existing research reports on synchronization of unweighted networks show that the small-world property enhances synchronization in small-world networks, such as the NW [2] and SW [1] networks, while although the heterogeneity may reduce the average path length (which improves the small-world property) but it weakens the synchronizability of scale-free networks [6].

Most studies on the synchronizability of networks are focused on unweighted and symmetrical networks. However, many real-world networks are weighted and asymmetrical; for example, social networks are typical weighted asymmetrical networks, in which the interaction between two individuals depends on several social factors such as age, social class, personal leadership, and charisma [7]. Nevertheless, there are some studies on the synchronization of weighted networks [8–10]. In particular, Hwang *et al.* [10] studied the synchronizability of weighted growing scale-free networks with normalized and asymmetrical coupling matrices. Their study shows that synchronization is enhanced when the couplings from older to younger nodes become dominant. With a normalized coupling matrix, the analysis of the synchronizability of such a network is not confined with the network topology and size. This result fits some real-world networks quite well, such as neuronal networks with node input not scaled with the number of connections. But the normalization of the coupling matrix leads all in degrees of nodes to be equal to 1, which means that every node in the network receives the same amount of information in a time interval. So, weighted networks with normalized coupling matrices are not suitable for those real-world networks with different information-receiving abilities in their nodes. A typical example is a computer network with computers as nodes and physical wires as links, in which some servers receive more information than the others in a time interval. It has been found that normalization of the coupling matrix can greatly enhance the synchronization of a network, but how the asymmetry of the coupling matrix affects the synchronizability of a network remains to be seen.

Considering the limitations of a weighted network with a normalized coupling matrix, as discussed above, in this paper the synchronizability of weighted aging scale-free networks with non-normalized and asymmetrical coupling matrices is studied. It will be shown that the synchronizability of a network is improved when the couplings from older to younger nodes become dominant, where the out degrees are heterogeneous and their in degrees are homogeneous, and that the synchronizability is seriously weakened or even lost when the couplings from younger to older nodes become dominant, where the out degrees are homogeneous and their in degrees are heterogeneous. It should be noted that the latter phenomenon cannot be seen in weighted growing scale-free networks with normalized and asymmetrical coupling matrices. It will show how the changes of the aging exponent or the average degree affect the synchronizability of such a network. It is found that some synchronous properties can be changed when an asymmetrical weighting parameter approaches a critical value, −1. For example, the heterogeneity of the degree distribution in a network can improve the synchronizability when the asymmetrical parameter approaches −1 and that a smaller average degree also can improve the synchronizability when the asymmetrical parameter approaches −1.

The rest of the paper is organized as follows. In Sec. II, the complex dynamical network model is described, a new weighting method is proposed, and the stability of the

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network synchronization is discussed. In Sec. III, the effects of the asymmetry of the coupling matrix on the network synchronizability are investigated, with detailed analysis and simulation results. Section IV reports the simulation results on a representative example of coupled Lorenz systems, with brief conclusions given in Sec. V.

II. STABILITY OF SYNCHRONIZATION IN COMPLEX DYNAMICAL NETWORKS

A. The complex dynamical network model and the weighting method

For the growing scale-free networks with aging sits introduced in [11], a node number is introduced according to the appearance order in the growing process, where a small node number corresponds to an old node and a large node number corresponds to a young node, and every link is weighted during the growing process.

Consider a network consisting of N linearly coupled identical systems, described by

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) - \sigma \sum_{j=1}^{N} G_{ij}\mathbf{H}(\mathbf{x_j}), \quad i = 1, \ldots, N, \tag{1}$$

where $\dot{\mathbf{x}}=\mathbf{F}(\mathbf{x})$ denotes the dynamics of the individual node, $\mathbf{H}(\mathbf{x})$ is a linear vector-valued function, σ is the coupling strength, $[G_{ij}]$ is a zero row-sum coupling matrix with off-diagonal entries $G_{ij}=-A_{ij}\Theta_{ij}$, where $\boldsymbol{A}$ is the adjacency matrix and $\Theta_{ij}=\frac{1-\theta}{2}\left(\Theta_{ij}=\frac{1+\theta}{2}\right)$ for $i>j(i<j)$, in which the parameter $-1<\theta<1$ governs the asymmetry of the coupling matrix in the network and the limit $\theta\to-1$ ($\theta\to1$) represents a unidirectional coupling where the old (young) nodes drive the young (old) ones.

According to [12], the in degree of a node i in the above weighted network is defined as

$$k_{i-in} = \sum_{\substack{j \\ j\neq i}} - G_{ij}, \tag{2}$$

and the out degree of a node j in the above weighted network is defined as

$$k_{j-out} = \sum_{\substack{i \\ i\neq j}} - G_{ij}. \tag{3}$$

The heterogeneity of the degrees in such a weighted network can be described by the covariance of the degree distribution, defined as

$$D = E[k - E(k)]^2, \tag{4}$$

where $E(k)$ is the mean value of all node degrees, which is equal to the number of the increased links after a new node is added into the network. Clearly, the larger the covariance D, the more heterogeneous the degree distribution.

Based on the weighting method proposed here, the mean values of in degree and out degree are equal; however, the covariances of the in degree and out degree are commonly unequal but they can be adjusted by the weighting parameter θ. In [10], the normalized weighting method is used, by setting $G_{ij}=-A_{ij}\frac{\Theta_{ij}}{\sum_{j\in K_i}\Theta_{ij}}$, where K_i is the set of k_i neighbors of the ith node. The weight of every link will be adjusted constantly with the addition of new nodes to the network. This weighting method results in that the in degree of each node in the weighted network remains the same and equals to 1, even not considering the asymmetry of the coupling matrix. The synchronizability of the weighted network will be improved greatly when the asymmetrical parameter $\theta=0$, so it is important to analyze how the asymmetry of the coupling matrix affects the synchronizability of the weighted network. With the proposed weighting method, for a chosen parameter θ, the weight of each link only relates to the ages of the two connected nodes but it is not adjusted with the increase of the number of nodes. It is easily implemented in a practical network, for example, in a circuit network, since a weight of a link can be the line resistance between the two connected circuits, and each existing resistance (the weight of a link) is not affected by a newly added circuit node. Both in-degree distribution and out-degree distribution can be adjusted by changing the asymmetrical parameter θ in the proposed weighting method, so that one can study the synchronizability of a weighted network for only two situations: (i) the in degrees are homogeneous and the out degrees are heterogeneous; (ii) the in degrees are heterogeneous and the out degrees are homogeneous.

B. Stable region of synchronous states in the network

Following the ideas of [13], the synchronizability of a network is inspected by the linear stability of the synchronous states ($\mathbf{x}_i=\mathbf{x}_s, \forall i$). By diagonalizing the variational equations, one obtains N blocks in the form of

$$\xi_i = J\mathbf{F}(\mathbf{x}_s)\xi_i + \sigma\lambda_i\mathbf{H}(\xi_i) \quad i = 1, \ldots, N, \tag{5}$$

which differ only by the eigenvalues λ_i of the coupling matrix $\boldsymbol{G}$ (here, $J\mathbf{F}$ is the Jacobian matrix). The linear stability of the synchronization manifold is determined by the largest Lyapunov exponents of Eq. (5) associated with $v=\sigma\lambda_i$ (also called the master stability function, MSF [14]). If all the largest Lyapunov exponents associated with $\lambda_i(i\geq2)$ are negative, then the synchronization manifold associated with $\lambda_1=0$ is linearly stable.

For a generic θ, the coupling matrix here is asymmetric; therefore, its spectrum is contained in the complex plane ($\lambda_1=0;\lambda_l=\lambda_l^r+j\lambda_l^i, l=2,\ldots,N$). Moreover, since all elements of $\boldsymbol{G}$ are real, complex eigenvalues appear in conjugate pairs. Order all eigenvalues of $\boldsymbol{G}$ according to the order of their increasing real parts. Let $M=\max\{\mathrm{Im}(\lambda_l^i)\}$, $l=2,\ldots,N$, denote the largest imaginary part of the eigenvalues. According to Gerschgorin's circle theorem, all eigenvalues of $\boldsymbol{G}$ are contained within the union of circles (C_i) whose centers are the diagonal elements of $\boldsymbol{G}$ (namely, $k_{i\text{-}in}$) and radii are the sums of the absolute values of the other elements in the corresponding rows (namely, $\{\lambda_l\}\subset\cup_i C_i[k_{i-in},\sum_{j\neq i}|G_{ij}|]$).

Let S be the bounded region in the complex plane where the master stability function provides a negative Lyapunov exponent. When $\{\sigma\lambda_l, l=2,\ldots,N\}$ are fully contained in S

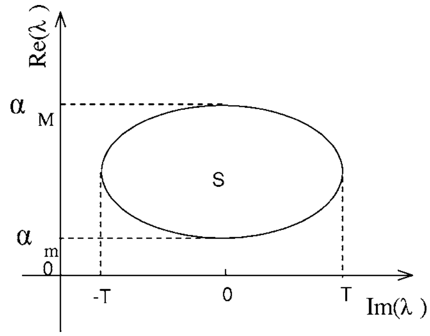

FIG. 1. The schematic diagram of a stable region.

for a given σ, the synchronous states of the network are stable. So, the synchronizability of the network is described by the range of σ, within which the network can achieve synchronization. The larger the range of σ, the better the synchronizability of the network. Since the shape and the size of the stable region S are irregular and they vary with the nodes and the linear vector-valued function $\mathbf{H}$, it is difficult to give an accurate description of the range of σ. Roughly, a schematic diagram of a stable region S of a typical network is shown in Fig. 1, where the nodes can be Lorenz systems or Rossler oscillators. When σ satisfies the following conditions:

$$\sigma \cdot M < T, \tag{6}$$

$$\alpha_m < \sigma\lambda_2^r < \alpha_M; \quad \alpha_m < \sigma\lambda_N^r < \alpha_M, \tag{7}$$

one may say that $\{\sigma\lambda_l, l=2,\ldots,N\}$ is entirely contained in the stable region S. The range of σ derived from Eqs. (6) and (7) is as follows:

$$\sigma \in \left\{\frac{\alpha_m}{\lambda_2^r}, \min\left(\frac{T}{M}, \frac{\alpha_M}{\lambda_N^r}\right)\right\}. \tag{8}$$

In Eq. (8): (1) if $\frac{T}{M} > \frac{\alpha_M}{\lambda_N^r}$, then $\frac{\lambda_N^r}{M} > \frac{\alpha_M}{T}$; therefore, $\sigma \in \left\{\frac{\alpha_m}{\lambda_2^r}, \frac{\alpha_M}{\lambda_N^r}\right\}$, so that the larger the λ_2^r, the smaller the λ_N^r, namely, the smaller the $\frac{\lambda_N^r}{\lambda_2^r}$, the better the synchronizability; (2) if $\frac{T}{M} < \frac{\alpha_M}{\lambda_N^r}$, then $\frac{\lambda_N^r}{M} < \frac{\alpha_M}{T}$; therefore, $\sigma \in \left\{\frac{\alpha_m}{\lambda_2^r}, \frac{T}{M}\right\}$, so that the larger the λ_2^r, the smaller the M, the better the synchronizability. Since $\frac{\alpha_M}{T}$ is determined by the node dynamics and the linear vector-valued function $\mathbf{H}$, the synchronizability of a weighted network should be estimated by considering both the node dynamics and the linear vector-valued function $\mathbf{H}$, as well as the coupling matrix.

Of course, if $\frac{\lambda_N^r}{\lambda_2^r}$ and M take smaller values at the same time in a weighted network, then the synchronizability of the network will be improved, independent of the node dynamics and the vector-valued function $\mathbf{H}$. But our studies show that the largest imaginary value M often takes a larger value when $\frac{\lambda_N^r}{\lambda_2^r}$ takes a smaller value in a weighted network with an asymmetrical coupling matrix. So, it is necessary to consider the node dynamics and the linear vector-valued function $\mathbf{H}$.

Now, consider a dynamical network with the Lorenz system as its nodes, described by

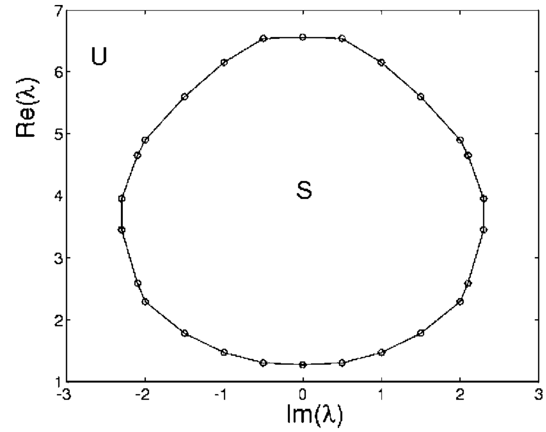

FIG. 2. Stable region of synchronous states in the coupled Lorenz network.

$$\dot{x} = 10(y - x),$$

$$\dot{y} = 23x - y - xz,$$

$$\dot{z} = xy - z \tag{9}$$

which is chaotic with a double-scrolls attractor [15].

For this network, the linear vector-valued function $\mathbf{H} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, a stable region of the synchronous states calculated by the MSF method is shown in Fig. 2. The three parameters corresponding to Fig. 1 are $\alpha_M=6.56$, $\alpha_m=1.27$, $T=2.3$, so $\frac{\alpha_M}{T}=2.852$.

III. EFFECTS OF THE ASYMMETRY OF THE COUPLING MATRIX ON THE SYNCHRONIZABILITY

In this section, the question how the asymmetry in the coupling matrix affects the synchronizability of a weighted network is addressed.

The network model here is a growing scale-free network with aging sites, proposed in [11]. Start from $m+1$ fully connected network. At each step, a new node is added with m links, connecting to old nodes with probability p_i defined by

$$p_i = \frac{k_i \tau_i^{-\alpha}}{\sum_j k_j \tau_j^{-\alpha}}, \tag{10}$$

where τ_i is the age of node i and α is the aging exponent and k_i is the number of connections to node i. The relationship between α and the degree distribution exponent γ [namely, $P(k) \propto k^{-\gamma}$, for large k] has been studied in [11], showing that γ varies from 2 to 3 with α varying from $-\infty$ to 0. When $\alpha=0$, the network becomes the original BA growing scale-free network [4]; when $-2 < \alpha < 1$, the smaller the α, the smaller the γ, that is, the smaller the α, the more heterogeneous the network.

Using an aging scale-free network with $N=500$, $m=5$, and $\alpha=0$, we have studied the relationships between some

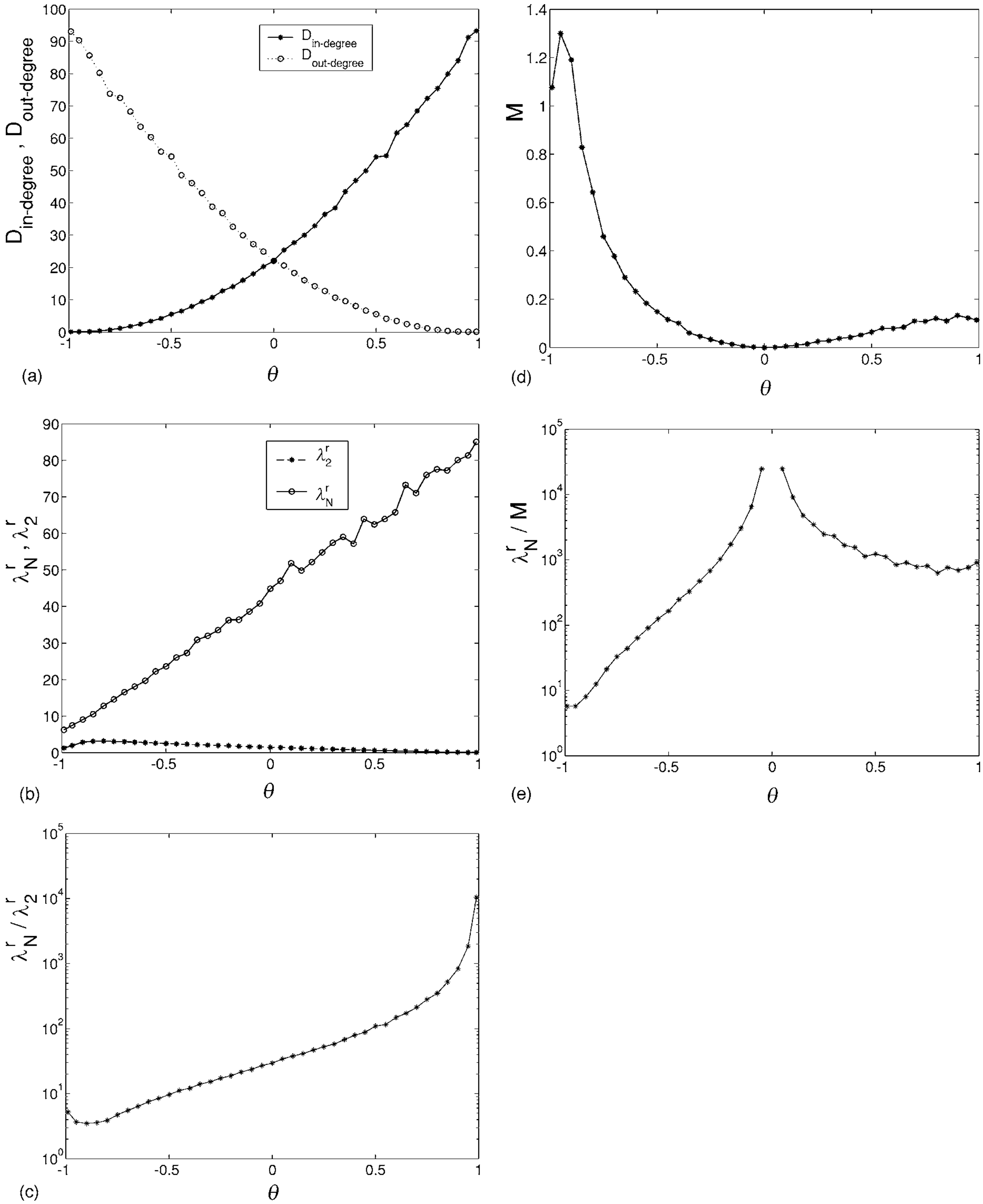

FIG. 3. Effects of the weighting parameter θ on the synchronizability of a growing scale-free network with $\alpha=0$, $N=500$, $m=5$; (a) $D_{\text{out-degree}}$ and $D_{\text{in-degree}}$ vs θ; (b) λ_2^r, λ_N^r vs θ; (c) $\frac{\lambda_N^r}{\lambda_2^r}$ vs θ; (d) M vs θ; (e) $\frac{\lambda_N^r}{M}$ vs θ.

given parameters and the asymmetrical weighting parameter θ, as shown in Fig. 3, where $D_{\text{out-degree}}$ is the out degree covariance, $D_{\text{in-degree}}$ is the in degree covariance, λ_2^r and λ_N^r are the second and the Nth real parts of eigenvalues of the matrix $\boldsymbol{G}$, respectively, M is the largest imaginary part of the eigenvalues of $\boldsymbol{G}$.

All calculation results are averages over 20 different realizations of networks with $N=500$. According to Fig. 3, one can obtain the following results: the out degree covariance $D_{\text{out-degree}}$ becomes smaller and smaller, and the in degree covariance $D_{\text{in-degree}}$ becomes larger and larger, when the weighting parameter θ varies from -1 to $+1$; the real part of the second eigenvalue λ_2^r takes the largest value, the ratio $\frac{\lambda_N^r}{\lambda_2^r}$ takes the smallest value, and the largest imaginary part M takes the largest value, when θ takes a value about -0.9; with

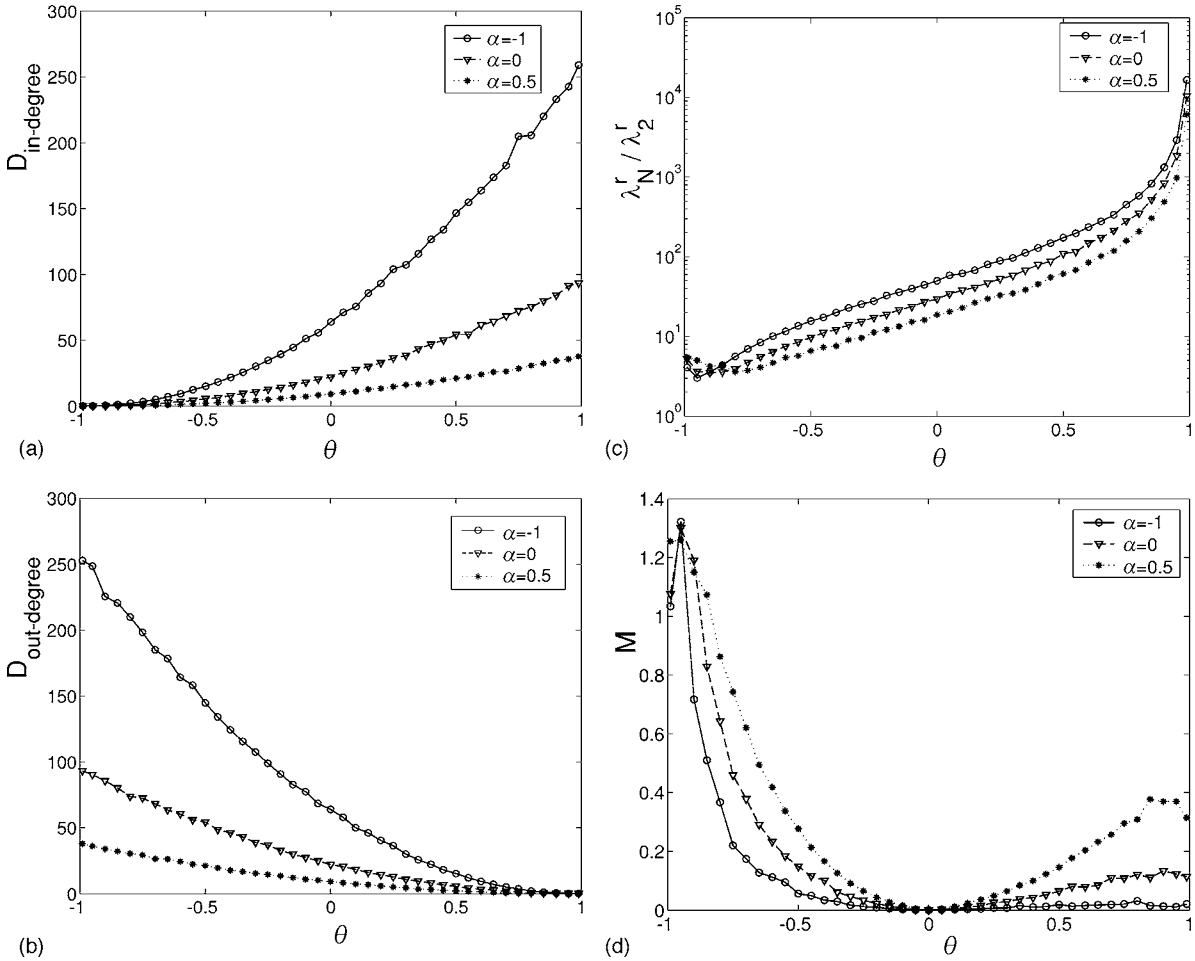

FIG. 4. Effects of the aging exponent α on the synchronizability in the aging scale-free network with N=500, m=5: (a) $D_{\text{in-degree}}$ vs θ; (b) $D_{\text{out-degree}}$ vs θ; (c) $\frac{\lambda_N^r}{\lambda_2^r}$ vs θ; (d) M vs θ.

the increase of θ, the ratio $\frac{\lambda_N^r}{\lambda_2^r}$ increases exponentially but M decreases to a smaller value.

It can also be seen from Fig. 3(e) that $\frac{\lambda_N^r}{M}\big|_{\min}=5.712$, and when the Lorenz system is taken as the node system with the linear vectorial function $H=\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$, one gets $\frac{\alpha_M}{T}=2.852$, so $\frac{\lambda_N^r}{M}>\frac{\alpha_M}{T}$ in the range of $-1<\theta<1$. According to the analysis on Eq. (8), the range of σ, which ensures the synchronization of the network, is $\sigma\in\left\{\frac{\alpha_m}{\lambda_2^r},\frac{\alpha_M}{\lambda_N^r}\right\}$, so the synchronizability of the network can be evaluated by examining the value of $\frac{\lambda_N^r}{\lambda_2^r}$. The best synchronizability can be achieved with θ being about -0.9, while $\frac{\lambda_N^r}{\lambda_2^r}$ takes the smallest value. The worst synchronizability appears when $\theta\to 1$, while $\frac{\lambda_N^r}{\lambda_2^r}$ approaches infinity. Since $\frac{\alpha_M}{\alpha_m}\in[5,100]$ for various chaotic oscillators [13], the network cannot achieve synchronization in the latter case. Actually, the network lost synchronizability with $\theta\geq 0.5$ while $\frac{\lambda_N^r}{\lambda_2^r}\geq 100$.

Figure 3(c) shows that the synchronizability of the network can be changed dramatically with adjusting the asymmetrical weighting parameter θ. The synchronizability is achieved when $\theta<0$, where the couplings from older to younger nodes become dominant, and is weakened or even lost with $\theta>0$ where the couplings from younger to older nodes become dominant.

At this point, we give some explanations about the above-observed phenomenon by using the in degree and out degree distributions. In the above-described weighting method, both in degrees and out degrees have the same mean value, which does not vary with the weighting parameter θ and is equal to the number of added links, m, when a new node joins the network. Whatever fixed state the node system stays, a periodic state or a chaotic state, as long as all the node systems are the same, the state average value of every node system is the same when they are uncoupled. The amount of information that a node received from other nodes is directly correlated with the in degree of the node, in a given time interval, so the larger the in degree, the more the information received. Meanwhile, the amount of information that a node is transmitted to other nodes is directly correlated with the out degree of the node, in a given time interval, so the larger the out degree, the more the information will be transmitted. When θ=0, the coupling matrix is symmetrical, both in degrees and out degrees have the same power-law distribution,

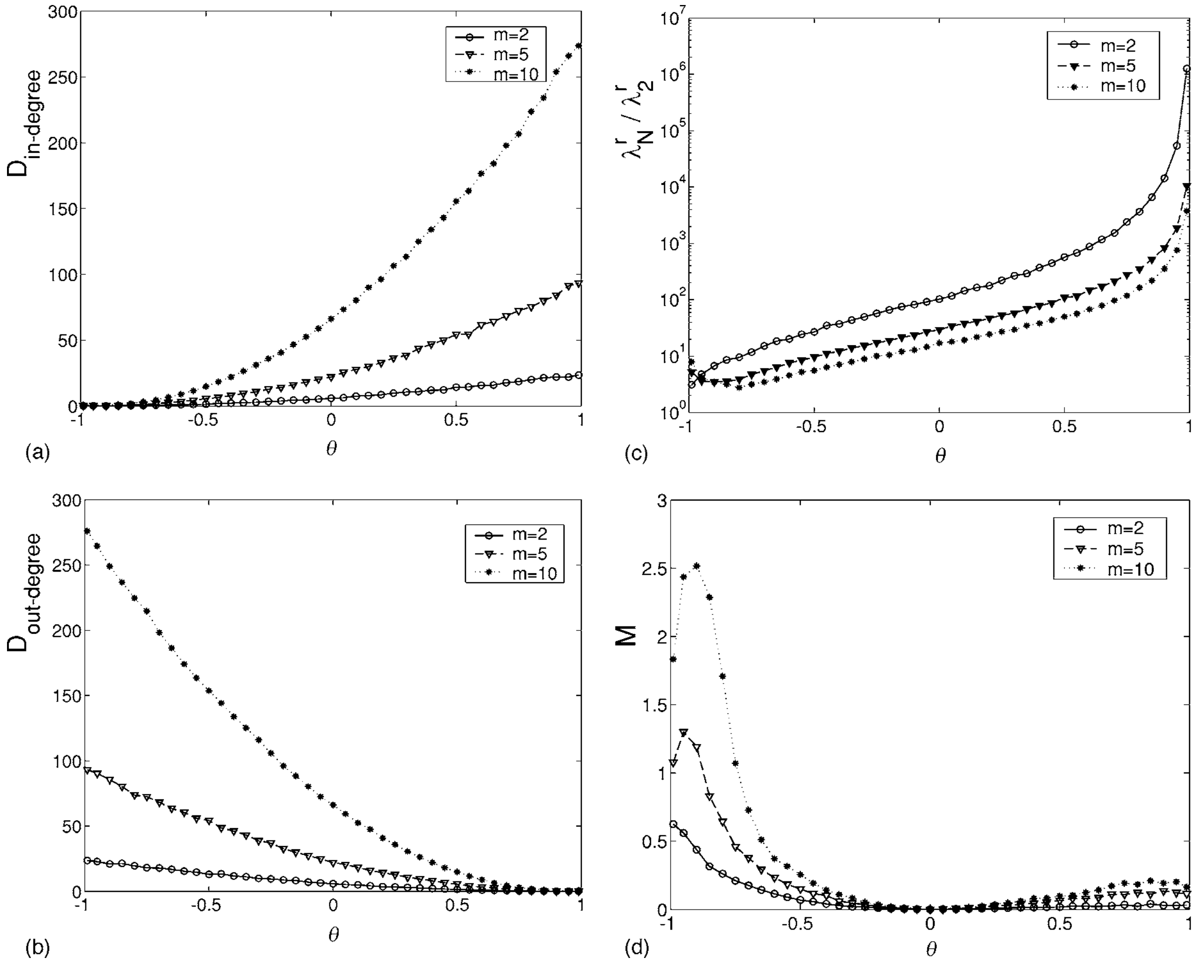

FIG. 5. Effects of the average degree on the synchronizability of weighted scale-free networks with $N=500$, $\alpha=0$: (a) $D_{\text{in-degree}}$ vs θ with $m=2$, 5, 10; (b) $D_{\text{out-degree}}$ vs θ with $m=2$, 5, 10; (c) $\frac{\lambda_N^r}{\lambda_2^r}$ vs θ with $m=2$, 5, 10; (d) M vs θ with $m=2$, 5, 10.

so the in degrees and out degrees have the same covariance. When θ approaches -1, the out degree covariance becomes larger and the in degree covariance becomes smaller, which means that the out degrees of nodes are heterogeneous. In this case, a few nodes have large out degrees and they transmit large amounts of information. At the same time, a few nodes have small out degrees, so they transmitted very little information; and the in degrees of nodes are homogeneous, so each node receives approximately the same amount of information. Since the total amounts of information transmitted by all nodes are equal to those received by all nodes in the network, the information received by most nodes comes from a few nodes with large out degrees; therefore, the network achieves synchronization easily. On the contrary, when θ approaches $+1$, the in degree covariance becomes larger and the out degree covariance becomes smaller; that is, the in degrees of nodes are heterogeneous and the out degrees of nodes are homogeneous. In this case, the amounts of information transmitted by different nodes are approximately equal to each other but the amounts of information received by different nodes are very different from each other. A few nodes with large in-degrees receive large amounts of information, which come from many different nodes. These information signals are very different in both altitudes and phases, so they may cancel each other, at least partially, inducing noneffective communications. This is similar to the case where surrounding nodes drive the center node in a star-

TABLE I. Different ranges of the coupling strength σ with various values of θ in Lorenz dynamical networks with $N=500$, $m=5$, $\alpha=0$.

θ	–0.99	–0.95	–0.9	–0.85	–0.8	–0.5	0	0.2	0.5	0.8
$\bar{\lambda}_2^r$	1.2246	2.0794	2.6555	3.0261	3.1036	2.4987	1.4496	1.1021	0.6068	0.2045
$\bar{\lambda}_N^r$	6.1540	7.3020	9.0635	10.8920	12.6586	24.1352	40.9585	53.3261	63.9404	74.9560
σ	(1.0371, 1.066)	(0.6108, 0.8984)	(0.4783, 0.7238)	(0.4197, 0.6023)	(0.4092, 0.5182)	—	—	—	—	—

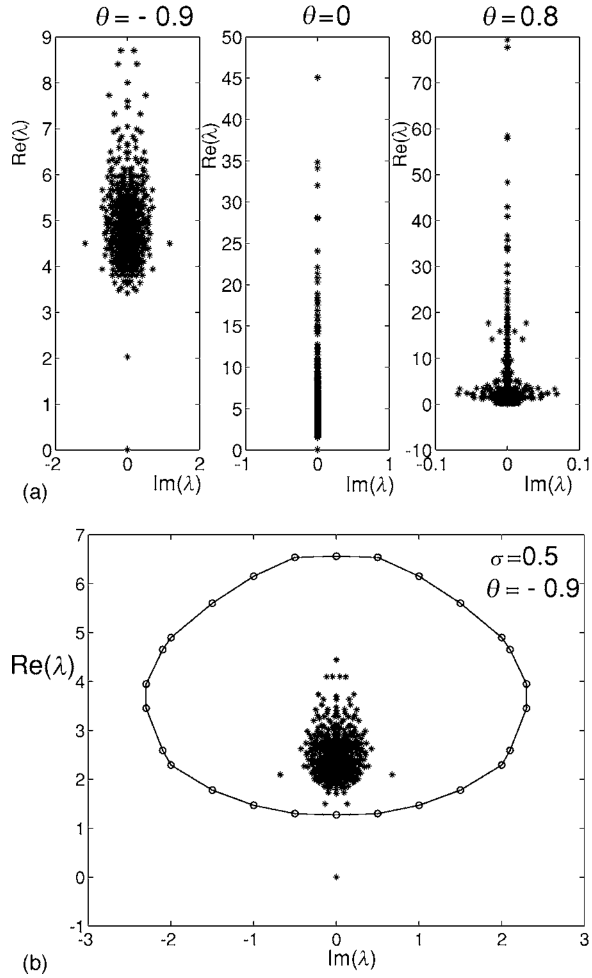

FIG. 6. Simulation study of coupled Lorenz dynamical networks: (a) three different distributions of eigenvalues of the coupling matrix $\boldsymbol{G}$ with $\theta=-0.9$, 0.0, 0.8, respectively; (b) relationship between the stable region of the coupled Lorenz systems and the points $\{\sigma^{*}\lambda_l, l=1,2,3,\ldots,N\}$ with $\theta=-0.9$ and $\sigma=0.5$.

shaped coupled network, so that the synchronizability is weakened dramatically or even lost.

Next, we further studied the effect of the aging exponent α on the synchronizability of weighted aging scale-free networks. The synchronizability of the networks varies with the weighting parameter θ, as seen from Figs. 4(c) and 4(d), where the three curves correspond to $\alpha=-1$, $\alpha=0$, and $\alpha=0.5$, respectively. Again, all calculations are the averages over 20 different realizations of networks with $N=500$ and $m=5$. Both the in degree covariance and the out degree covariance increase with the decrease of α, which are shown in Figs. 4(a) and 4(b), respectively, so the heterogeneity in such networks is enhanced with the decrease of α. Figure 4(c) shows that the smaller the aging parameter α, the smaller the value of $\frac{\lambda_N^r}{\lambda_2^r}$, with θ in the range of $(-1,-0.9)$. When the Lorenz system is used as the nodes and its state variable z is taken as the coupling variable, the inequality $\frac{\lambda_N^r}{M} > \frac{\alpha_M}{T}$ is satisfied in the range of $-1<\theta<1$, so the synchronizability of the network can be evaluated by examining the value of $\frac{\lambda_N^r}{\lambda_2^r}$. It follows that the heterogeneity of the network improves the synchronizability when $-1<\theta<-0.9$, but weakens the synchronizability when $-0.9<\theta<1$.

Recall that for unweighted networks with symmetrical coupling matrices and a given power-law degree distribution exponent γ, the larger the average degree, the smaller the average path length, the better the synchronizability.

In the following, we studied the effect of the average degree on the synchronizability in weighted scale-free networks with $N=500$ and $\alpha=0$. Figure 5 shows three curves corresponding to $m=2$, 5, 10, respectively. All calculation results are the averages over 20 different realizations of networks with $N=500$ and $\alpha=0$. When the Lorenz system is taken as the node system and its state variable z is taken as the coupling variable, the inequality $\frac{\lambda_N^r}{M} > \frac{\alpha_M}{T}$ is satisfied in the range of $-1<\theta<1$, so the synchronizability of the networks can be evaluated by examining the value of $\frac{\lambda_N^r}{\lambda_2^r}$. In the range of $-1<\theta<-0.95$, the smaller the average degree m, the smaller the value of $\frac{\lambda_N^r}{\lambda_2^r}$. It then follows that a smaller average degree improves the synchronizability when $-1<\theta<-0.95$ but it weakens the synchronizability when $-0.95<\theta<+1$, which is different from the result reported in [10], which studies the effect of the average degree on the synchronizability in the weighted growing scale-free networks with normalized asymmetrical coupling matrices. In [10], it shows that the larger the average degree, the better the synchronizability in the whole range of $-1<\theta<1$.

We have also studied the effect of the average degree on the synchronizability of the weighted networks with the aging exponent $\alpha\neq 0$, obtaining the same result as the case of $\alpha=0$.

IV. SIMULATION STUDY

In this section, we report our studies on the synchronizability of weighted growing scale-free networks with the Lorenz system as its nodes and the state variable z as the coupling variable.

The equations of the Lorenz system shown in Eq. (9) and the aging scale-free networks with $N=500$, $m=5$, $\alpha=0$ are used. Table I gives the different ranges of the coupling strength σ corresponding to various values of the asymmetrical parameter θ, where $\bar{\lambda}_2^r$ and $\bar{\lambda}_N^r$ are the average value of λ_2^r and the average value of λ_N^r, respectively, over 20 different realizations of networks. The range of the coupling strength σ with a given θ is obtained from Eq. (8).

It can be seen from Table I that the better synchronization is achieved with θ near the value of -0.9, where the range of the coupling strength σ is a relatively bigger one. The results shown in Table I are consistent with that shown in Fig. 3.

Figure 6(a) shows the distributions of the eigenvalues of the coupling matrix $\boldsymbol{G}$ in three different cases of $\theta=-0.9$, 0.0, and 0.8, respectively. One can see that the eigenvalues expand quickly along the direction of the real axis with the increase of θ, which is consistent with the calculation results shown in Fig. 3. Figure 6(b) shows the stable region of the Lorenz systems coupled through its variable z and the points $\{\sigma\lambda_l, l=1,\ldots,N\}$ with the coupling strength $\sigma=0.5$ and the asymmetrical parameter $\theta=-0.9$. It can be seen that all the points $\{\sigma\lambda_l, l=2,\ldots,N\}$ except $\lambda_1=0$ are contained in the stable region of the synchronous states of the coupled Lorenz systems, clearly indicating that the Lorenz dynamical network achieves synchronization.

V. CONCLUSIONS

In this paper, the synchronizability of weighted aging scale-free networks with non-normalized asymmetrical coupling matrices has been studied in some detail. The synchronizability of such weighted networks can be dramatically affected by the asymmetrical parameter θ. Some new results, different from earlier reports, were obtained and analyzed. The synchronizability of the weighted networks can be improved when the couplings from older to younger nodes become dominant, where the out degrees of nodes are heterogeneous and their in degrees are homogeneous, and the synchronizability can be seriously weakened or even lost when the couplings from younger to older nodes become dominant, where the in-degrees of nodes are heterogeneous and their out degrees of nodes are homogeneous. As the asymmetrical parameter θ approaches the critical value -1, the smaller the aging exponent, the better the synchronizability; otherwise, the bigger the aging exponent, the better the synchronizability. Similarly, a smaller average degree can also improve the synchronizability when θ approaches -1.

ACKNOWLEDGMENT

This work was supported by the National Nature Science Foundation of China under Grant No. 70571017.

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